Chapter 12 Notes

The Binomial Distribution
STP-231

Arizona State University

Discrete Probability Distributions

- The **probability distribution** of a random variable X tells us what values X can take and how to assign probabilities to those values
- In the discrete case, all possible values (and subsequently their probabilities) can be listed
- Let the random variable X be the number of heads when tossing a coin twice (Bernoulli)
- Let the random variable Y be the number of heads when tossing a coin n times (Binomial)

Discrete Probability Distributions
Continued

- For probabilities on intervals of possible values we take note of which outcomes are defined in the interval
- Note, that unlike the continuous case $\Pr(a < X < b)$ may not be the same $\Pr(a \leq X \leq b)$ The same is true for other combinations of inclusion
- For example, let Y be the count of heads that come up after flipping a coin 10 times
- Then $\Pr(Y > 8) \neq \Pr(Y \geq 8)$

Bernoulli Random Variables

- The simplest type of random variable
- Can take on two values: “success” or “failure”
- We define p to be the probability of success; meaning the probability of failure is $1 - p$. A single coin toss is an example
- success and failure defined arbitrarily. A tail could be success/failure depending on how you define it

Binomial Distribution Definition

- Assume a series of n independent Bernoulli experiments is conducted
- Each independent repetition of the experiment is a trial
- Each experiment results in either "success" or "failure"
- n is the number of trials, k the number of successes, and $n - k$ is the number of failures
- The probability of success is the same for each experiment, regardless of the outcomes of the other trials (i.e. independence)

The Binomial Coefficient

- Say we have n total items and we want to choose k. Let's say we have 5 total letters to choose from (A, B, C, D, E) and choose 3. For our first choice, we have 5 options, then for the second we have 4, then 3. The number of ways to choose these k items in conjunction would the product of the previous arrangement, i.e.

$$5 \cdot 4 \cdot 3 = \frac{5!}{2!} \text{ This is True if order matters}$$

- This is not quite right however, because we double count many arrangements. For example

$$\text{A, B, C} = \text{C, B, A} = \text{B, A, C}$$

because we choose the same letters each time. Therefore, we have to divide by all possible "repeat arrangements", which is $3 \cdot 2 \cdot 1 = 3! = 6$ in this example.

Binomial Coefficient

- Therefore, we have

$$\binom{5}{3} = \frac{5!}{(5-3)!(3)!} = 10$$

- In general:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

The Binomial Distribution

- The probability distribution for the number of successes in a sequence of Bernoulli trials
- For a binomial random variable Y, the probability that the n trials results in k successes and $n-k$ failures is given by:

$$\Pr(Y=k) = \Pr(k \text{ successes}) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad (1)$$

- The distribution reflects the probabilities of the k successes and the $n-k$ failures
- Multiplied by the unique number of ways that those n trials can be arranged

Example

Describe the distribution of 2 coin tosses

- Let's say tails is a success, and heads is a failure. Instead of counting out the outcomes, lets let Y be the number of tails

$$\Pr(Y=0) = \binom{2}{0} \cdot p^0 \cdot (1-p)^{2-0} = (1-p)^2$$

$$\Pr(Y=1) = \binom{2}{1} \cdot p^1 \cdot (1-p)^{2-1} = 2 \cdot p \cdot (1-p)$$

$$\Pr(Y=2) = \binom{2}{2} \cdot p^2 \cdot (1-p)^{2-2} = p^2$$

- If $p = 0.5$, i.e. a fair coin, what are the probabilities?

Conditions to satisfy for a Binomial RV

- Each trial only has two possible outcomes: success or failure
- Trials are independent of each other
- The number of trials is fixed at n
- The probability of success, denoted by p, is the same for all trials

Sampling Distribution of a Count

- Choose and SRS (simple random sample) of size n from a population with proportion p successes
- Recall: the probability for randomly choosing an individual with a certain characteristic is equivalent to the population proportion (relative frequency) of individuals with that characteristic in the population
- When the population is much larger than the sample, the count X of successes in the sample has approximately the binomial distribution with parameters n and p

Example

Assume we have one mole of air in a room (6.02×10^{23} air particles). Suppose we split the room in half. Then they are 6.02×10^{23} ways to have all but 1 of the 6.02×10^{23} air particles on one side of the room. This would almost surely suffocate any living thing on the 1-particle side of the room. And there are so many ways for this to happen!
So why don't we spontaneously choke more often? Because there are WAY more ways to have equilibrium. This is the idea of entropy in physics

Example

Ladies Home Journal magazine in 1993 that 66% of all dog owners greet their dog before greeting their spouse when they return home at the end of the workday. Suppose that 12 dog owners are selected at random.

(a) What is the probability exactly 4 dog owners greet their dog before greeting their spouse?

(b) What is the probability between 3 and 5 (including 3 and 5) dog owners greet their dog before their spouse?

(c) What is the probability at least 1 dog owner greets their dog before greeting their spouse?

(d) What is the probability that 50% of these dog owners greet their dog before greeting their spouse?

Solutions

Let X be the R.V. describing how many dog owners greet their dogs, with a success being greeting their dog first

(a)

$$\begin{aligned}\Pr(X = 4) &= \frac{n!}{k!(n-k!)}p^k(1-p)^{n-k} \\ &= \frac{12!}{4!(12-4)!}(0.66)^4(1-0.66)^{12-4} \\ &= 0.0168\end{aligned}$$

(b)

$$\begin{aligned}\Pr(3 \le X \le 5) &= \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5) \\ &= 0.00384 + 0.0168 + 0.0521 \\ &= 0.0727\end{aligned}$$

Solutions
Continued

(c)

$$\begin{aligned}\Pr(X \leq 1) &= 1 - \Pr(X = 0) \\ &= 1 - \frac{12!}{0!(12-0)!}(0.66)^0(1-0.66)^{12-0} \\ &= 1 - 0.000002386 \\ &= 0.999997614\end{aligned}$$

(d) 50% of dog owners means number of successes is 6

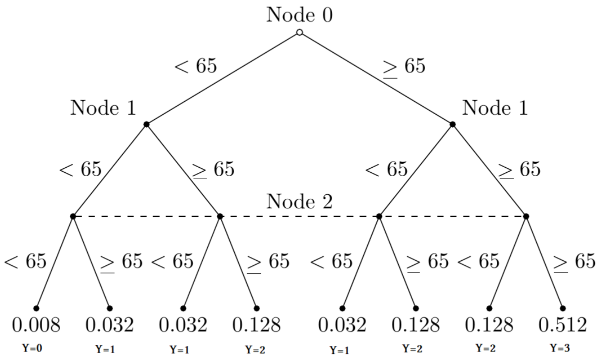
$$\begin{aligned}\Pr(X = 6) &= \frac{12!}{6!(12-6)!}(0.66)^6(1-0.66)^{12-6} \\ &= 0.11798\end{aligned}$$

Example

U.S. National Center for Health Statistics states that there is an 80% that a person aged 20 will be alive at age 65. If we randomly select 3 people, and Y is the event that we select a person who is alive at 65, we can create a tree diagram to determine the possible outcomes.

- Determine the probability for each outcome using the results for the tree diagram.
- Use the Binomial distribution formula to show how we can get the same results
- Find the probability that at least one person out of the three was alive at age 65
- Find $\Pr(Y > 2)$ and $\Pr(Y < 1)$

The tree



There are multiple ways to get $Y = 1$ or $Y = 2$ (3 of each). We must sum these probabilities

The Binomial Distribution

$$\Pr(Y = 0) = \binom{3}{0} 0.8^0 (1 - 0.8)^{3-0} = 0.2^3 = 0.008$$

$$\Pr(Y = 1) = \binom{3}{1} 0.8^1 (1 - 0.8)^{3-1} = 3 \cdot 0.8 \cdot 0.2^2 = 0.096$$

$$\Pr(Y = 2) = \binom{3}{2} 0.8^2 (1 - 0.8)^{3-2} = 3 \cdot 0.8^2 \cdot 0.2 = 0.384$$

$$\Pr(Y = 3) = \binom{3}{3} 0.8^3 (1 - 0.8)^{3-3} = 0.8^3 = 0.512$$

- Make sure they sum to 1! A little bit easier to use the formula, but not quite as clear

Which Event is a Success?

Suppose a quality control inspector selects a random sample of 10 motherboards from a very large shipment for inspection. Unknown to the inspector, only 60% of the motherboards meet specifications. What is the probability that no more than 1 of the 10 motherboards in the sample fails the inspection?

- Say X is the R.V. which describes the number of motherboards that fail to meet specifications, i.e. failing to meet specifications is the success, i.e.

$$\Pr(X = k) = \binom{10}{k} 0.4^k (0.6)^{10-k}$$

- Y could also be the $\#$ that do meet specs, i.e.

$$\Pr(Y = k) = \binom{10}{k} 0.6^k (0.4)^{10-k}$$

Mean and Variance of Binomial

- The mean is
$$\mu = np \tag{2}$$
- The variance is
$$\sigma^2 = np(1-p)$$
- What are the mean and variance from the Ladies Home Journal magazine dog example?